UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
Computer Engineering

ECE 204 *Numerical methods*

Taylor series

Douglas Wilhelm Harder, LEL, M.Math.
dwharder@uwaterloo.ca
dwharder@gmail.com

Introduction

- In this topic, we will
  - Review Taylor series as seen from calculus
  - Introduce a variation on the formula appropriate for this course
  - Look at the error term
  - Observe that the formula tells us exactly how much the error drops
  - Introduce big-O notation to describe the error

The derivative

- There are different ways of representing the derivative:

$$f'(x), f''(x), f'''(x)$$

$$\dot{u}(t), \ddot{u}(t), \dddot{u}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x), \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x), \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x)$$

  – In this course, we will use a variation of the first:

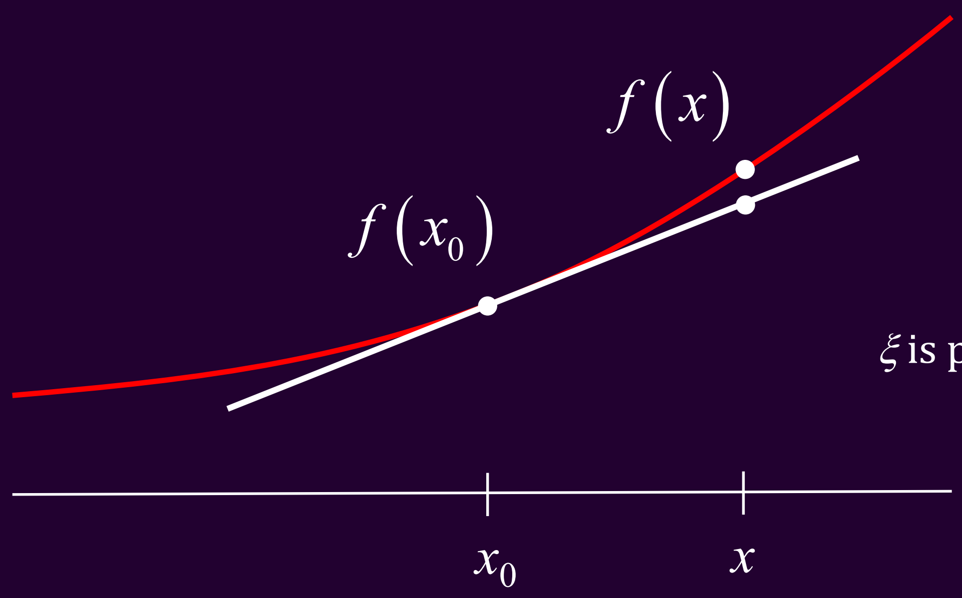
$$f^{(1)}(x), f^{(2)}(x), f^{(3)}(x), \ldots$$

Taylor series

- Given a function where the second derivative is continuous, we know that

$$f(x) \approx f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{1}{2} f^{(2)}(\xi_0)(x - x_0)^2$$

$$x_0 \le \xi_0 \le x$$

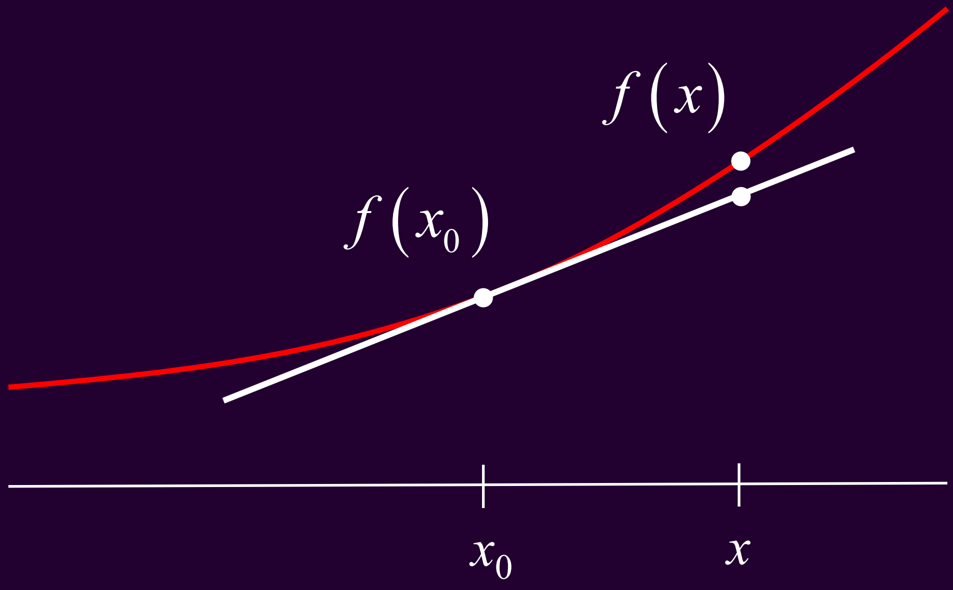


ξ is pronounced *k-sigh* or *k-see*

Taylor series

- A quick aid-memoire for $x - x_0$ versus $x_0 - x$:

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{1}{2} f^{(2)}(\xi_0)(x - x_0)^2$$



Taylor series

- Mathematicians are generally interested in general solutions:
  - This equation is true for all values of x

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{1}{2} f^{(2)}(\xi_0)(x - x_0)^2$$

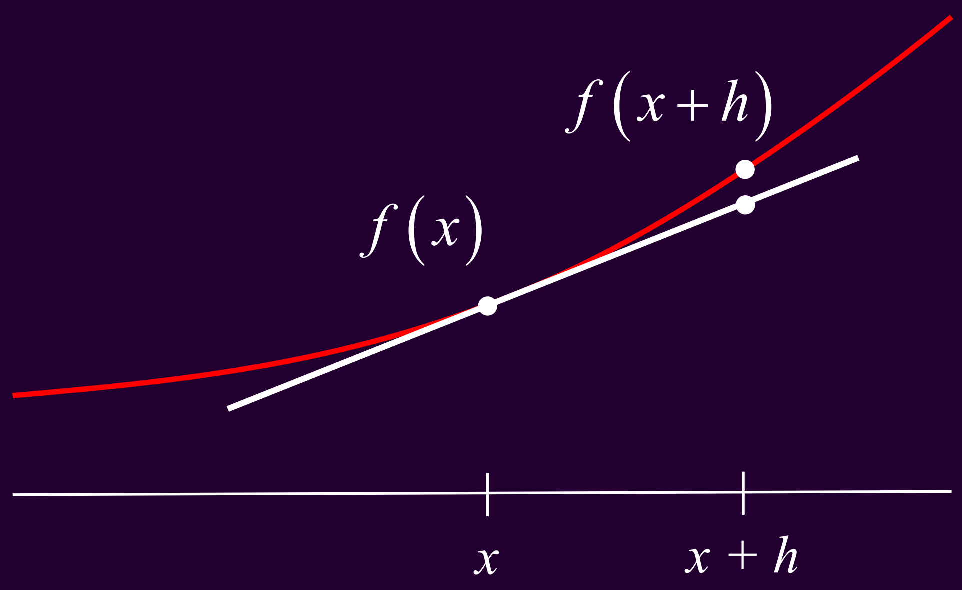
- Engineers, however, are generally only interested in points close to the known value x
  - Thus, instead of having two different values of x and x_0, we will focus on a point x and approximate the value at $x + h$

Taylor series

- Thus, our approach will be as follows:

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(\xi)h^2$$

$$x \le \xi \le x+h$$



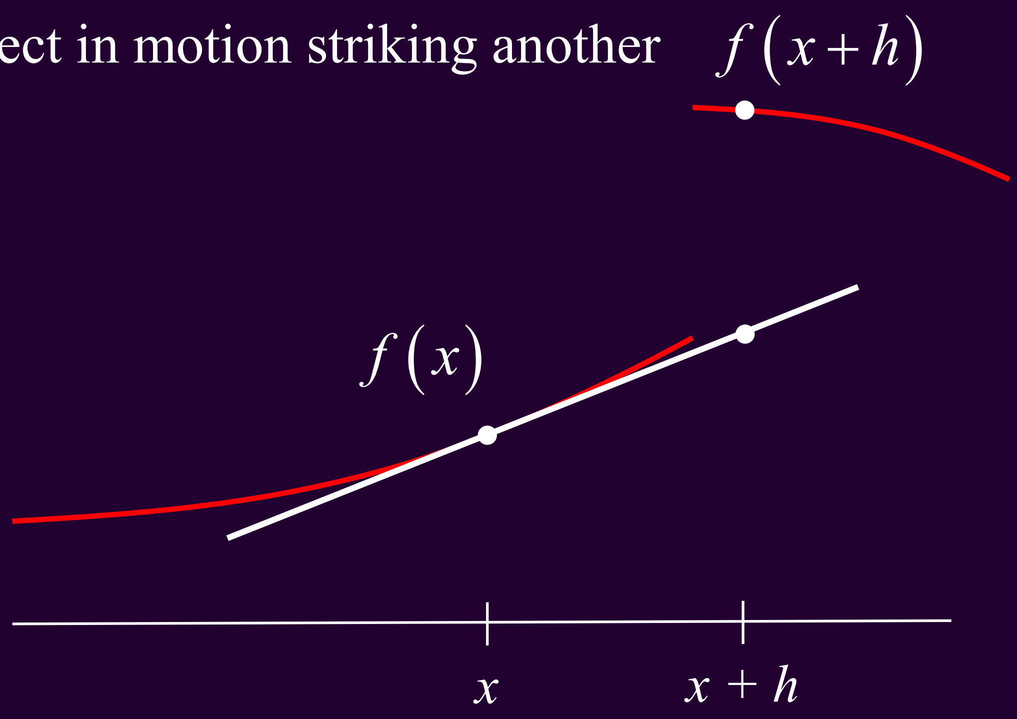
Taylor series

- Thus, we have
  - The exact value
  - The approximation
  - The error

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(\xi)h^2$$

Taylor series

- Note that the Taylor series is not useful if the function is not sufficiently differentiable:
  - Flipping a switch in an electrical circuit
  - An object in motion striking another



Taylor series

- Taylor series will generally be used to estimate the error of other techniques we will use
  - The error will be in terms of the absolute error, not relative

$$\frac{1}{2}\left|f^{(2)}(\xi)\right||h|^2$$

- When we begin find approximations to solutions to initial-value problems,

  Taylor series will be the basis of our formulas

Taylor series

- On occasion, we will use higher order Taylor series

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(x)h^2 + \frac{1}{3!}f^{(3)}(\xi)h^3$$

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(x)h^2 + \frac{1}{3!}f^{(3)}(x)h^3 + \frac{1}{4!}f^{(4)}(\xi)h^4$$

- The unknown for the error term will always be the Greek equivalent of the variable we are dealing with

x and ξ

t and τ

τ is pronounced *tau*, rhyming with *cow*

- To keep life simple, we will use
  - f for functions of x
  - y for functions of t

Big Oh

- You will recall that the error was

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(\xi)h^2$$

  – Assuming that the second derivative is approximately constant, the only way to make the error smaller is to reduce h

  – Thus, if we divide h by two, the error should go down by a factor of approximately four
  – Also, if we multiply h by two, the error should go up by a factor of approximately four

Big Oh

- Suppose we want to approximate $\sin(2 + h)$

n	$h = 2^{-n}$	$\sin(2 + h)$	Approximation	Absolute error	$\frac{1}{2}\left\|\sin(2)\right\| h^2$	Ratio
1	0.5	0.5984721441	0.7012240086	0.1028	0.1137	
2	0.25	0.7780731969	0.8052607177	0.02719	0.02842	0.2646
3	0.125	0.8503197898	0.8572790723	0.006959	0.007104	0.2560
4	0.00625	0.8815297858	0.8832882495	0.001758	0.001776	0.2527
5	0.003125	0.8958509980	0.8962928382	0.0004418	0.0004440	0.2513
6	0.0015625	0.9026844011	0.9027951325	0.0001107	0.0001110	0.2506
7	0.00078125	0.9060185633	0.9060462797	0.00002772	0.00002775	0.2503
8	0.000390625	0.9076649200	0.9076718532	0.000006933	0.000006937	0.2502
9	0.0001953125	0.9084829062	0.9084846400	0.000001734	0.000001734	0.2501
10	0.00009765625	0.9088905999	0.9088910334	0.0000004335	0.0000004336	0.2500

Big Oh

- Consequently, we will describe the error of a first-order Taylor series approximation as O(h^2)

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(\xi)h^2$$

  - Throughout this course, we will find different algorithms to approximate solutions to numerical problems
  - In many cases, there will be a parameter h that we can adjust, and the error will depend on that parameter
    - We will see algorithms that are O(h), O(h^2), O(h^5), etc.

Summary

- Following this topic, you now
  - Are aware of Taylor series with respect to $f(x + h)$
  - Understand the part that Taylor series will play in this course
  - Know that the Taylor formula gives the absolute error
    - If the appropriate derivative is continuous, we have a good estimate as to what the error is
  - We have also seen that the 1st-order Taylor series approximation is what we will describe as $O(h^2)$

References

[1] https://en.wikipedia.org/wiki/Taylor_series

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

Disclaimer

These slides are provided for the ECE 204 *Numerical methods* course taught at the University of Waterloo. The material in it reflects the author's best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.